The descent statistic on involutions is not log-concave

Marilena Barnabei, Flavio Bonetti, and Matteo Silimbani [*]

Abstract. We establish a combinatorial connection between the sequence $(i_{n,k})$ counting the involutions on n letters with k descents and the sequence $(a_{n,k})$ enumerating the semistandard Young tableaux on n cells with k symbols. This allows us to show that the sequences $(i_{n,k})$ are not log-concave for some values of n, hence answering a conjecture due to F. Brenti.

1   Introduction

A finite sequence $(s_1, \ldots, s_n)$ of real numbers is said to be *unimodal* if there exists an index t such that $s_1 \leq s_2 \leq \cdots \leq s_t$ and $s_t \geq \cdots \geq s_{n-1} \geq s_n$. An arbitrary sequence $(s_i)_{i \in \mathbb{N}}$ is *log-concave* if $s_{i-1} \cdot s_{i+1} \leq s_i^2$ for every $i > 0$. It is immediately seen that a finite log-concave sequence of positive numbers is unimodal. In recent years, several authors focused on the study of such two properties in relation to the distribution of the descent statistic on involutions. More precisely, given a word $w = w_1 \ldots w_n$ on a linearly ordered alphabet, the *descent set* of w is defined as $\mathrm{des}(w) = \{1 \leq i < n : w_i \geq w_{i+1}\}$ and the cardinality of the set $\mathrm{des}(w)$ is denoted by $d(w)$. An analogous definition can be given for the *ascent set* of a word. If σ is an involution, the descent set of σ is the descent set of the word $\sigma(1) \ldots \sigma(n)$. Let $i_{n,k}$ be the

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[*]Dipartimento di Matematica - Università di Bologna

number of involutions on n letters with k descents and let

$$I_n(x) = \sum_{k=0}^{n-1} i_{n,k} x^k$$

be the generating function of the sequence $i_{n,k}$, for every $n \in \mathbb{N}$. Strehl [16] proved that the coefficients of $I_n(x)$ are symmetric. Recently, Brenti (see [6]) conjectured that the coefficients of the polynomial $I_n(x)$ are log-concave. Dukes [6] obtained some partial results on the unimodality of such coefficients and Guo and Zeng [8] succeeded in proving that the sequence $i_{n,k}$ is unimodal.

In the present note, we disprove Brenti's conjecture exploiting the combinatorial relation of the sequence $i_{n,k}$ with the sequence $a_{n,s}$ counting semistandard Young tableaux on n cells with s symbols. This last sequence is easily seen to be not log-concave. We show that the generating functions of these two sequences are related by a binomial transformation. This fact allows us to refute the log-concavity of the polynomial $I_n(x)$. Moreover, we deduce an explicit formula for the integers $i_{n,k}$.

The relation between standard and semistandard Young tableaux sheds new light on the combinatorial properties of Young tableaux. For example, it provides an immediate proof of the well known fact that every Schur function s_λ can be expressed as a sum of suitable fundamental quasi-symmetric functions. Moreover, the present techniques allow to investigate other properties of the distribution of the descent statistic both on the set of involutions itself [2] and on some notable subset of involutions [1].

2   Standard Young Tableaux

Consider the set $\mathscr{T}_n$ of standard Young tableaux on n cells. It is well known that the Robinson-Schensted algorithm establishes a bijection $\psi : \mathscr{I}_n \to \mathscr{T}_n$, where $\mathscr{I}_n$ is the set of involutions over $[n] := \{1, 2, \ldots, n\}$. A further bijective map χ exists between $\mathscr{T}_n$ and the set $\mathscr{Y}_n$ of reverse Yamanouchi words of length n. We recall that a *reverse Yamanouchi word* is a word w with integer entries such that any left subword of w does not contain more occurrences of the symbol $(i + 1)$ than of i, for every $i \geq 1$. The Yamanouchi word $\chi(T)$ associated to a given tableau T is obtained by placing in the i-th position

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the row index of the cell of T containing the symbol i. For example, if T is the standard Young tableau

$$T = \begin{array}{ccc} 1 & 3 & 5 \\ 2 & 6 & 7 \\ 4 & 8 & \end{array}$$

we have $\chi(T) = 1\,2\,1\,3\,1\,2\,2\,3$.

Clearly, the composition $\varphi := \chi \circ \psi$ yields a bijection between the sets $\mathscr{I}_n$ and $\mathscr{Y}_n$. We remark that φ turns each ascent of a given involution σ into a descent of the correspondent reverse Yamanouchi word.

Let $i_{n,h}$ be the number of involutions $\sigma \in \mathscr{I}_n$ with h descents and $y_{n,k}$ the number of reverse Yamanouchi words of length n with k descents. The preceding remark implies that

$$y_{n,k} = i_{n,n-1-k}.$$

The present approach leads to an immediate proof of the following result originally due to Strehl [16]:

Proposition 1 *For every $n \in \mathbb{N}$, we have*

$$y_{n,k} = y_{n,n-1-k}.$$

Proof Given a reverse Yamanouchi word y, consider the conjugate word $\tilde{y}$ defined as follows: if $y_i = m$, then $\tilde{y}_i$ is the number of occurrences of the integer m in the left subword $y_1 \ldots y_i$. For example, the conjugate of the word

$$y = 1\,2\,1\,3\,1\,2\,2\,3$$

is

$$\tilde{y} = 1\,1\,2\,1\,3\,2\,3\,2$$

Note that, if y is associated with the tableau T, $\tilde{y}$ is associated with the conjugate tableau of T. Clearly, y has k descents if and only if $\tilde{y}$ has $n-1-k$ descents.

$\diamond$

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For every $n \in \mathbb{N}$, define

$$I_n(x) = \sum_{\sigma \in \mathscr{I}_n} x^{d(\sigma)} = \sum_{h=0}^{n-1} i_{n,h} x^h.$$

This polynomial can be rewritten in terms of reverse Yamanouchi words as follows:

$$I_n(x) = \sum_{k=0}^{n-1} y_{n,n-1-k} x^k = \sum_{k=0}^{n-1} y_{n,k} x^k = \sum_{y \in \mathscr{Y}_n} x^{d(y)}.$$

3 Semistandard Young Tableaux

Given a Ferrers diagram λ, a *semistandard Young tableau* on k symbols of shape λ is an array obtained by placing into each cell of the diagram an integer in $[k]$ so that the entries are strictly increasing by rows and weakly increasing by columns. We consider the infinite matrix $A = (a_{n,k})$, with $n, k \in \mathbb{N}$, where $a_{n,k}$ denotes the number of semistandard Young tableaux with n cells and k symbols. An explicit expression for the column generating function $F_k(x)$ of the matrix A

$$F_k(x) = \sum_{n \geq 0} a_{n,k} x^n = \frac{1}{(1-x)^k (1-x^2)^{\binom{k}{2}}}$$

was firstly given by Schur (see [10]). This yields immediately the following explicit formula for the integers $a_{n,k}$:

$$a_{n,k} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{\frac{k(k-1)}{2} + j - 1}{j} \binom{k + n - 2j - 1}{k - 1}. \tag{1}$$

The following properties of the row sequences of the matrix A are direct consequences:

Proposition 2 *The sequence $(a_{n,k})_{k \in \mathbb{N}}$ is in general not log-concave.*

Proof Exploiting Formula (1), we have:

$$a_{45,2}^2 = 304704 < 307970 = a_{45,1} \cdot a_{45,3}.$$

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We are now interested in establishing a connection between the sequences $(a_{n,k})$ and $(y_{n,k})$. To this aim, we associate with a given semistandard tableau T a biword (w, y) as follows: w contains all the entries in T listed in non-decreasing order. The word y is obtained by listing the row indices of the occurrences of each symbol, starting from the smallest one. If a symbol j occurs more than once, we write the corresponding row indices in increasing order. It is easy to check that y is a Yamanounchi word. Note that the biword (w, y) uniquely determines the tableau T. In fact, applying to the biword (w, y) the Robinson-Schensted-Knuth column insertion procedure, we get the pair (Y, T), where Y is a row Yamanouchi tableau, namely, a tableau whose i-th row consists only of letters i for all i.

We are now going to show that the number of semistandard tableaux on s symbols associated with a given reverse Yamanouchi word y depends only on the number of descents of y. Fix a reverse Yamanouchi word y with k descents. Any semistandard tableau with associated biword (w, y) must contain at least $k + 1$ different symbols. In fact, if y has a descent at position i, by the definition of the correspondence between tableaux and biwords the integers w_i and w_{i+1} must be different. This implies that the set of tableaux T with s symbols and associated word y corresponds bijectively to the set of words w with $1 \leq w_1 \leq w_2 \leq \cdots \leq w_n \leq s$, where the inequalities are strict in correspondence of the descents of y. Every such word w is uniquely determined by the sequence $\delta := w_1 - 1, w_2 - w_1, \ldots, w_n - w_{n-1}, s - w_n$, which is a composition of the integer $s - 1$ such that its i-th component δ_i is at least one whenever y has a descent at the i-th position. For this reason, we can consider the word δ' defined as follows:

$$\delta_i' = \begin{cases} \delta_i - 1 & \text{if } y \text{ has a descent at the i-th position} \\ \delta_i & \text{otherwise} \end{cases},$$

which is, of course, a composition of the integer $s - k - 1$.

For example the semistandard tableau on 5 symbols

$$T = \begin{array}{lll} 1 & 2 & 3 \\ 2 & 3 & \\ 4 & & \\ 5 & & \end{array}$$

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is associated to the Yamanouchi word $y = 1\,1\,2\,1\,2\,3\,4$ with descents at positions 1 and 3. In this case, we have:

$$w = 1\,2\,2\,3\,3\,4\,5$$

$$\delta = 0\,1\,0\,1\,0\,1\,1\,0$$

$$\delta' = 0\,0\,0\,0\,0\,1\,1\,0.$$

We are now in position to prove the following:

Theorem 3 *The total number of semistandard Young tableaux with n cells and k symbols is*

$$a_{n,s} = \sum_{k=0}^{s-1} \binom{n+k}{k} y_{n,s-k-1} \tag{2}$$

and conversely,

$$y_{n,k} = \sum_{j=1}^{k+1} (-1)^{k-j+1} \binom{n+1}{k-j+1} a_{n,j}. \tag{3}$$

Proof The preceding observations show that the semistandard tableaux with s symbols and associated word y are in bijection with the compositions of the integer $s-k-1$ into $n+1$ parts. In other terms, the number of semistandard Young tableaux with s symbols whose associated reverse Yamanouchi word y has k descents is

$$\binom{n+s-k-1}{n}.$$

Formula (2) follows directly by these considerations. The second identity can be easily obtained by inversion.

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Combining formulae (1) and (3) we get an explicit expression for the integers $y_{n,k}$:

Corollary 4 *The number $y_{n,k}$ of reverse Yamanouchi words of length n with k descents is*

$$y_{n,k} = \sum_{j=1}^{k+1} (-1)^{k-j+1} \binom{n+1}{k-j+1} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{\binom{j}{2}+i-1}{i} \binom{n+j+2i-1}{j-1}. \tag{4}$$

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$\diamond$

We remark that Formula (2) implies the following relation between the generating function $A_n(x) = \sum_{k \geq 0} a_{n,k} x^k$ of the n-th row of the matrix A and the polynomial $I_n(x)$:

Theorem 5 *We have:*

$$A_n(x) = \frac{x I_n(x)}{(1-x)^{n+1}}.$$

$\diamond$

In conclusion of this section, we submit that Theorem 7.19.7 in [13] can be rephrased as an immediate consequence of the described correspondence between reverse Yamanouchi words and semistandard tableaux. In fact, let λ be a partition of the integer n and let $Y(\lambda)$ be the set of Yamanouchi words whose associated standard tableau has shape λ. For every $y \in Y(\lambda)$, we denote by $S(y)$ the set of semistandard tableaux associated with y. The fundamental quasi-symmetric function L_y can be defined as:

$$L_y(x_1, \ldots, x_m) = \sum_{\substack{1 \leq i_1 \leq \cdots \leq i_n \leq m \\ i_j < i_{j+1} \text{ if } j \in des(y)}} x_{i_1} \cdots x_{i_n}.$$

Then, the Schur function $s_\lambda(x_1, \ldots, x_m)$ can be expressed in terms of fundamental quasi-symmetric functions as follows:

$$s_\lambda(x_1, \ldots, x_m) = \sum_{\substack{S \ semistandard \\ sh(S) = \lambda}} x^{w(S)} = \sum_{y \in Y(\lambda)} \sum_{S \in S(y)} x^{w(S)} =$$

$$= \sum_{y \in Y(\lambda)} \sum_{\substack{1 \leq i_1 \leq \cdots \leq i_n \leq m \\ i_j < i_{j+1} \text{ if } j \in des(y)}} x_{i_1} \cdots x_{i_n} = \sum_{y \in Y(\lambda)} L_y,$$

where $w(S)$ is the content of S.

4 Disproof of the conjecture

First of all, we recall a general result appearing in [9]:

Proposition 6 *The product $p(x) \cdot q(x)$ of a unimodal polynomial $p(x)$ and a log-concave polynomial $q(x)$ is unimodal. If $p(x)$ is log-concave, the product $p(x) \cdot q(x)$ is log-concave as well.*

The relation between the sequences $(y_{n,k})$ and $(a_{n,k})$ described in Theorem 3 allows us to refute the log-concavity of the polynomials $I_n(x)$. In fact, we have:

Theorem 7 *The polynomials $I_n(x)$ are in general not log-concave.*

Proof Formula (2) shows that the polynomial

$$p_n(x) = \sum_{k=0}^{n} a_{n,k} x^k$$

is the product of the two polynomials $I_n(x)$ and

$$q_n(x) = \sum_{k=0}^{n} \binom{n+k-1}{k} x^k.$$

The polynomial $q_n(x)$ is log-concave. In fact, the condition

$$\binom{n+k-2}{k-1}\binom{n+k}{k+1} \leq \binom{n+k-1}{k}$$

is equivalent to

$$\frac{n+k}{k+1} \leq \frac{n+k-1}{k}$$

that holds for every $n \geq 1$. Hence the log-concavity of $I_n(x)$ would imply the log-concavity of the polynomial $p_n(x)$, contradicting Proposition 2.

$$\diamond$$

In fact, exploiting Formula (4), we get, for instance

$$y_{50,1}^2 = 390625 < 465570 = y_{50,0} \cdot y_{50,2}.$$

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